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Milne's hypergeometric functions in terms of free fermions

A Yu Orlov[1,2] and D M Scherbin[2]

[1] Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-85-02, Japan
[2] Nonlinear Wave Processes Laboratory, Oceanology Institute, 36 Nakhimovskii prospekt, Moscow 117851, Russia

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Abstract

We present the fermionic representation for the q-deformed hypergeometric functions related to Schur polynomials. We show that these multivariate hypergeometric functions are tau-functions of the KP hierarchy, and at the same time they are the ratios of Toda lattice tau-functions, considered by Takasaki, evaluated at certain values of higher Toda lattice times. The variables of the hypergeometric functions are related to the higher times of those hierarchies via a Miwa change of variables. The discrete Toda lattice variable shifts parameters of hypergeometric functions. Hypergeometric functions of type ${}_p\Phi_s$ can also be viewed as a group 2-cocycle for the ΨDO on the circle (the group times are higher times of TL hierarchy and the arguments of a hypergeometric function). We obtain the determinant representation and the integral representation of a special type of KP tau-functions, these results generalize some of the results of Milne concerning multivariate hypergeometric functions. We write down a system of partial differential equations for these tau-functions (string equations).

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1. Milne's hypergeometric series

1.1. Schur symmetric function

For a partition

$$\boldsymbol{n} = (n_1, n_2, \dots, n_r) \qquad n_1 \geqslant n_2 \geqslant \dots \geqslant n_r \qquad r \leqslant |\boldsymbol{n}| = n_1 + \dots + n_r \tag{1.1}$$

whose length $l(\boldsymbol{n}) = r \leqslant N$, and indeterminates $\boldsymbol{x}_{(N)} = (x_1, \dots, x_N)$, the Schur polynomial $s_{\boldsymbol{n}}(\boldsymbol{x}_{(N)})$, a symmetric function of variables $\boldsymbol{x}_{(N)}$, is defined as follows [12]:

$$s_{\boldsymbol{n}}(\boldsymbol{x}_{(N)}) = \frac{a_{\boldsymbol{n}+\delta}}{a_\delta} \qquad a_{\boldsymbol{n}} = \det(x_i^{n_j})_{1 \leqslant i,j \leqslant N} \qquad \delta = (N-1, N-2, \dots, 1, 0) \tag{1.2}$$

$$s_{\boldsymbol{m}}(\boldsymbol{x}_{(N)}) = 0 \qquad \text{if} \quad l(\boldsymbol{m}) > N. \tag{1.3}$$

In the KP theory it is suitable to use another definition of the Schur function corresponding to the partition $\boldsymbol{n} = (n_1, \ldots, n_r)$:

$$s_{\boldsymbol{n}}(\boldsymbol{t}) = \det(p_{n_i - i + j}(\boldsymbol{t}))_{1 \leqslant i,j \leqslant r} \tag{1.4}$$

where $p_m(\boldsymbol{t})$ is the elementary Schur polynomial defined by the Taylor expansion

$$\mathrm{e}^{\xi(\boldsymbol{t},z)} = \exp\left(\sum_{k=1}^{+\infty} t_k z^k\right) = \sum_{n=0}^{+\infty} z^n p_n(\boldsymbol{t}). \tag{1.5}$$

It is related to $s_{\boldsymbol{n}}(\boldsymbol{x}_{(N)})$ and $s_{\boldsymbol{n}'}(\boldsymbol{x}_{(N)})$, where a partition $\boldsymbol{n}'$ is conjugate to $\boldsymbol{n}$, as follows [12]:

$$s_{\boldsymbol{n}}(\boldsymbol{t}^+(\boldsymbol{x}_{(N)})) = s_{\boldsymbol{n}}(\boldsymbol{x}_{(N)}) \qquad s_{\boldsymbol{n}}(\boldsymbol{t}^-(\boldsymbol{x}_{(N)})) = s_{\boldsymbol{n}'}(\boldsymbol{x}_{(N)}) \tag{1.6}$$

via the following changes of variables (which is known as the Miwa change of variables in the literature on integrable systems):

$$t_m^+(\boldsymbol{x}_{(N)}) = \sum_{i=1}^{N} \frac{x_i^m}{m} \qquad t_m^-(\boldsymbol{x}_{(N)}) = -\sum_{i=1}^{N} \frac{x_i^m}{m}. \tag{1.7}$$

1.2. The multiple basic hypergeometric series related to Schur polynomials

There are several well known different multivariate generalizations of hypergeometric series of one variable [10, 11]. The multiple basic hypergeometric series related to Schur polynomials were introduced by Milne [1] as follows:

$${}_p\Phi_s\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_s \end{matrix}\middle| q, \boldsymbol{x}_{(N)}\right) = \sum_{\substack{\boldsymbol{n} \\ l(\boldsymbol{n}) \leqslant N}} \frac{(q^{a_1}; q)_{\boldsymbol{n}} \cdots (q^{a_p}; q)_{\boldsymbol{n}}}{(q^{b_1}; q)_{\boldsymbol{n}} \cdots (q^{b_s}; q)_{\boldsymbol{n}}} \frac{q^{n(\boldsymbol{n})}}{H_{\boldsymbol{n}}(q)} s_{\boldsymbol{n}}(\boldsymbol{x}_{(N)}) \qquad |q| < 1. \tag{1.8}$$

The coefficient $(q^a; q)_{\boldsymbol{n}}$ associated with partition $\boldsymbol{n}$:

$$(q^a; q)_{\boldsymbol{n}} = (q^a; q)_{n_1} (q^{a-1}; q)_{n_2} \cdots (q^{a-r+1}; q)_{n_r} \tag{1.9}$$

$$(q^a; q)_0 = 1 \qquad (q^a; q)_n = (1 - q^a)(1 - q^{a+1}) \cdots (1 - q^{a+n-1}). \tag{1.10}$$

The multiple $q^{n(\boldsymbol{n})}$ defined on the partition $\boldsymbol{n}$ and the q-deformed 'hook polynomial' $H_{\boldsymbol{n}}(q)$:

$$q^{n(\boldsymbol{n})} = q^{\sum_{i=1}^{l(\boldsymbol{n})}(i-1)n_i} \qquad H_{\boldsymbol{n}}(q) = \prod_{(i,j) \in \boldsymbol{n}} \left(1 - q^{h_{ij}}\right) \qquad h_{ij} = (n_i + n_j' - i - j + 1). \tag{1.11}$$

For $N = 1$ we obtain

$${}_p\Phi_s\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_s \end{matrix}\middle| q, x\right) = \sum_{n=0}^{\infty} \frac{(q^{a_1}; q)_n \cdots (q^{a_p}; q)_n}{(q^{b_1}; q)_n \cdots (q^{b_s}; q)_n} \frac{x^n}{(q; q)_n}. \tag{1.12}$$

For the bosonic representation of basic hypergeometric functions of one variable, see [14].

Many special functions and polynomials (such as the q-Askey–Wilson polynomials, q-Jacobi polynomials, q-Gegenbauer polynomials, q-Racah polynomials, q-Hahn polynomials, expressions for Clebsch–Gordan coefficients) are just $N = 1$ hypergeometric functions evaluated at special parameter values.

1.3. Hypergeometric series of a double set of arguments

$$
{}_p\Phi_s\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_s\end{matrix}\middle| q,\boldsymbol{x}_{(N)},\boldsymbol{y}_{(N)}\right) = \sum_{\substack{\boldsymbol{n}\\ l(\boldsymbol{n})\leqslant N}} \frac{(q^{a_1};q)_{\boldsymbol{n}}\cdots(q^{a_p};q)_{\boldsymbol{n}}}{(q^{b_1};q)_{\boldsymbol{n}}\cdots(q^{b_s};q)_{\boldsymbol{n}}}\frac{q^{n(\boldsymbol{n})}}{H_{\boldsymbol{n}}(q)}\frac{s_{\boldsymbol{n}}(\boldsymbol{x}_{(N)})s_{\boldsymbol{n}}(\boldsymbol{y}_{(N)})}{s_{\boldsymbol{n}}\left(1,q,q^2,\ldots,q^{N-1}\right)} \qquad |q|<1. \tag{1.13}
$$

This formula defines the multiple basic hypergeometric function of two sets of variables which was also introduced by Milne (see [1, 10]). For $q \to 1$ the functions (1.8) and (1.13) are also known as hypergeometric functions of the matrix argument which are related to zonal spherical polynomials for $GL(N, C)/U(N)$ symmetric space.

2. A brief introduction to the fermionic description of the KP and TL hierarchies [5, 9]

2.1. Fermionic operators and Fock space

We have fermionic fields

$$
\psi(z) = \sum_k \psi_k z^k \qquad \psi^*(z) = \sum_k \psi_k^* z^{-k-1} \tag{2.1}
$$

where fermionic operators satisfy the canonical anticommutation relations

$$
[\psi_m, \psi_n]_+ = [\psi_m^*, \psi_n^*]_+ = 0 \qquad [\psi_m, \psi_n^*]_+ = \delta_{mn}. \tag{2.2}
$$

Let us introduce left and right vacuums by the properties

$$
\psi_m|0\rangle = 0 \quad (m<0) \qquad \psi_m^*|0\rangle = 0 \quad (m\geqslant 0) \tag{2.3}
$$

$$
\langle 0|\psi_m = 0 \quad (m\geqslant 0) \qquad \langle 0|\psi_m^* = 0 \quad (m<0). \tag{2.4}
$$

The vacuum expectation value is defined by relations

$$
\langle 0|1|0\rangle = 1 \qquad \langle 0|\psi_m\psi_m^*|0\rangle = 1 \qquad m<0 \qquad \langle 0|\psi_m^*\psi_m|0\rangle = 1 \qquad m\geqslant 0 \tag{2.5}
$$

$$
\langle 0|\psi_m\psi_n|0\rangle = \langle 0|\psi_m^*\psi_n^*|0\rangle = 0 \qquad \langle 0|\psi_m\psi_n^*|0\rangle = 0 \qquad m\neq n. \tag{2.6}
$$

Let us note that relations (2.2)–(2.6) are invariant under the following transformation:

$$
\psi_n \to \mathrm{e}^{-T_n}\psi_n \qquad \psi_n^* \to \mathrm{e}^{T_n}\psi_n^* \qquad (T_n \in C). \tag{2.7}
$$

Consider infinite matrices $(a_{ij})_{i,j\in Z}$ satisfying the condition: there exists an N such that $a_{ij} = 0$ for $|i-j| > N$. Let us take the set of linear combinations of quadratic elements $\sum a_{ij} : \psi_i\psi_j^* :$, where : : denotes normal ordering $: \psi_i\psi_j^* : \; = \psi_i\psi_j^* - \langle 0|\psi_i\psi_j^*|0\rangle$. These elements together with 1 span an infinite-dimensional Lie algebra $\widehat{gl}(\infty)$:

$$
\left[\sum a_{ij} : \psi_i\psi_j^* :, \sum b_{ij} : \psi_i\psi_j^* : \right] = \sum c_{ij} : \psi_i\psi_j^* : + c_0 \tag{2.8}
$$

$$
c_{ij} = \sum_k a_{ik}b_{kj} - \sum_k b_{ik}a_{kj} \qquad c_0 = \sum_{i<0,j\geqslant 0} a_{ij}b_{ji} - \sum_{i\geqslant 0,j<0} a_{ij}b_{ji}. \tag{2.9}
$$

Now we define the operator g which is an element of the group corresponding to the Lie algebra $\widehat{gl}(\infty)$:

$$
g\psi_n g^{-1} = \sum_m \psi_m a_{mn} \qquad g^{-1}\psi_n^* g = \sum_m a_{nm}\psi_m^*. \tag{2.10}
$$

2.2. The KP and Toda tau-functions

First we define the vacuum vectors labelled by the integer M:

$$|M\rangle = \Psi_M|0\rangle \qquad \Psi_M = \psi_{M-1}\cdots\psi_0 \quad (M>0) \qquad \Psi_M = \psi_M^*\cdots\psi_{-2}^*\psi_{-1}^* \quad (M<0)$$
$$\langle M| = \langle 0|\Psi_M^* \qquad \Psi_M^* = \psi_0^*\cdots\psi_{M-1}^* \quad (M>0) \qquad \Psi_M^* = \psi_{-1}\psi_{-2}\cdots\psi_M \quad (M<0). \tag{2.11}$$

The tau-function of the KP equation and the tau-function of the two-dimensional Toda lattice (TL) are sometimes defined as

$$\tau_{KP}(M,\boldsymbol{t}) = \langle M|\mathrm{e}^{H(\boldsymbol{t})}g|M\rangle \qquad \tau_{TL}(M,\boldsymbol{t},\boldsymbol{t}^*) = \langle M|\mathrm{e}^{H(\boldsymbol{t})}g\mathrm{e}^{H^*(\boldsymbol{t}^*)}|M\rangle. \tag{2.12}$$

According to [5] the integer M in (2.12) plays the role of a discrete Toda lattice variable. The times $\boldsymbol{t}=(t_1,t_2,\ldots)$ and $\boldsymbol{t}^*=(t_1^*,t_2^*,\ldots)$ are called higher Toda lattice times [5, 9]. The first times of this set t_1,t_2,t_3 are independent variables for the KP equation (2.18), which is the first non-trivial equation in the KP hierarchy. $H(\boldsymbol{t})$ and $H^*(\boldsymbol{t}^*)$ belong to the following $\widehat{gl}(\infty)$ Cartan subalgebras:

$$H(\boldsymbol{t}) = \sum_{n=1}^{+\infty} t_n H_n \qquad H^*(\boldsymbol{t}^*) = \sum_{n=1}^{+\infty} t_n^* H_{-n} \qquad H_n = \sum_{m=-\infty}^{+\infty} :\psi_m\psi_{m+n}^*:. \tag{2.13}$$

For the Hamiltonians we have Heisenberg algebra commutation relations:

$$[H_n,H_m] = n\delta_{m+n,0}. \tag{2.14}$$

The action of $\mathrm{e}^{H(\boldsymbol{t})}$ on the fermions ψ_i, ψ_i^* and on the fermionic fields $\psi(z)$, $\psi^*(z)$ is

$$\mathrm{e}^{H(\boldsymbol{t})}\psi_i\mathrm{e}^{-H(\boldsymbol{t})} = \sum_{n=0}^{+\infty} p_n(\boldsymbol{t})\psi_{i-n} \qquad \mathrm{e}^{H(\boldsymbol{t})}\psi_i^*\mathrm{e}^{-H(\boldsymbol{t})} = \sum_{n=0}^{+\infty} p_n(\boldsymbol{t})\psi_{i+n}^* \tag{2.15}$$

$$\mathrm{e}^{H(\boldsymbol{t})}\psi(z)\mathrm{e}^{-H(\boldsymbol{t})} = \psi(z)\mathrm{e}^{\xi(\boldsymbol{t},z)} \qquad \mathrm{e}^{H(\boldsymbol{t})}\psi^*(z)\mathrm{e}^{-H(\boldsymbol{t})} = \psi^*(z)\mathrm{e}^{-\xi(\boldsymbol{t},z)} \tag{2.16}$$

$$\mathrm{e}^{-H^*(\boldsymbol{t}^*)}\psi(z)\mathrm{e}^{H^*(\boldsymbol{t}^*)} = \psi(z)\mathrm{e}^{-\xi(\boldsymbol{t}^*,z^{-1})} \qquad \mathrm{e}^{-H^*(\boldsymbol{t}^*)}\psi^*(z)\mathrm{e}^{H^*(\boldsymbol{t}^*)} = \psi^*(z)\mathrm{e}^{\xi(\boldsymbol{t}^*,z^{-1})}. \tag{2.17}$$

In terms of tau-functions [5, 9] the KP equation and TL equation are

$$4\partial_{t_1}\partial_{t_3}u = \partial_{t_1}^4 u + 3\partial_{t_2}^2 u + 3\partial_{t_1}^2 u^2 \qquad u = 2\partial_{t_1}^2\log\tau_{KP} \tag{2.18}$$

$$\partial_{t_1}\partial_{t_1^*}\varphi_n = \mathrm{e}^{\varphi_{n+1}-\varphi_n} - \mathrm{e}^{\varphi_n-\varphi_{n-1}} \qquad \mathrm{e}^{\varphi_n} = \frac{\tau_{TL}(n+1,\boldsymbol{t},\boldsymbol{t}^*)}{\tau_{TL}(n,\boldsymbol{t},\boldsymbol{t}^*)}. \tag{2.19}$$

The KP equation, that originally served in plasma physics (see [7] for references), now plays a very important role both in modern physics and in mathematics. It was integrated by a dressing method in the paper by Zakharov and Shabat [4, 7]. The TL equation was first integrated in [8]. In the present paper we use the approach of [2, 3, 5, 9].

3. Hypergeometric functions related to Schur functions

3.1. KP tau-function $\tau_r(M,\boldsymbol{t},\beta)$

Let r be a function of one variable. We consider an Abelian subalgebra in $\widehat{gl}(\infty)$ formed by the set of fermionic operators A_k,

$$A_k = \sum_{n=-\infty}^{\infty} \psi_{n-k}^*\psi_n r(n)r(n-1)\cdots r(n-k+1) \qquad k=(1,2,\ldots)$$
$$A(\beta) = \sum_{n=1}^{\infty}\beta_n A_n. \tag{3.1}$$

$[A_m, A_k] = 0$ for each m, k. The fermionic operators A_k resemble Toda lattice Hamiltonians $-H_k^*$ (2.13), and coincide with them if $r(n) = 1, n \in Z$. $\beta = (\beta_1, \beta_2, \ldots)$ is a collection of independent variables. For a given function r and a partition $\boldsymbol{n} = (n_1, \ldots, n_k)$, we introduce the notation

$$r_{\boldsymbol{n}}(M) = \prod_{i=1}^{k} r(1-i+M)r(2-i+M)\cdots r(n_i-i+M) \qquad r_{\boldsymbol{0}}(M) = 1. \tag{3.2}$$

Let us consider the tau-function of the KP hierarchy

$$\tau_r(M, \boldsymbol{t}, \beta) := \langle M|\mathrm{e}^{H(\boldsymbol{t})}\mathrm{e}^{-A(\beta)}|M\rangle. \tag{3.3}$$

Using the Taylor expansions $\mathrm{e}^H = 1 + H + \cdots$ and $\mathrm{e}^{-A} = 1 - A + \cdots$ one can derive:

Proposition 1. *We have an expansion*

$$\tau_r(M, \boldsymbol{t}, \beta) = 1 + \sum_{n=1}^{+\infty} \sum_{|\boldsymbol{n}|=n} r_{\boldsymbol{n}}(M)\, s_{\boldsymbol{n}}(\boldsymbol{t})\, s_{\boldsymbol{n}}(\boldsymbol{\beta}). \tag{3.4}$$

We shall not consider the problem of convergence of this series.

Remark 1. The variables $M, \boldsymbol{t}$ play the role of KP higher times, β is a collection of group times for a commuting subalgebra of additional symmetries of KP (see remark 7 in [13]). From a different point of view (3.4) is a tau-function of the two-dimensional Toda lattice [9] with two sets of continuous variables $\boldsymbol{t}, \beta$ and one discrete variable M. Formula (3.4) is symmetric with respect to $\boldsymbol{t} \leftrightarrow \beta$. This 'duality' supplies us with the string equations [3] which characterize a tau-function of hypergeometric type (see below). In [2] the expansions of a tau-function in terms of Schur functions were considered, without specifying the coefficients and in a different context.

Now we introduce the operators

$$\tilde{A}_k = \sum_{n=-\infty}^{+\infty} \psi_n \psi_{n+k}^* \tilde{r}(n+1)\ldots\tilde{r}(n+k) \qquad k = (1, 2, \ldots) \qquad \tilde{A}(\tilde{\beta}) = \sum_{n=1}^{\infty} \tilde{\beta}_n \tilde{A}_n. \tag{3.5}$$

Proposition 2. *The generalization of proposition 1*

$$\langle M|\mathrm{e}^{\tilde{A}(\tilde{\beta})}\mathrm{e}^{-A(\beta)}|M\rangle = 1 + \sum_{n=1}^{+\infty} \sum_{|\boldsymbol{n}|=n} (\tilde{r}r)_{\boldsymbol{n}}(M) s_{\boldsymbol{n}}(\tilde{\beta}) s_{\boldsymbol{n}}(\beta). \tag{3.6}$$

In what follows one can put $\tilde{r} = 1$, since (3.6) depends only on $\tilde{r}r$.

3.2. $H_0(\boldsymbol{T})$, twisted fermions $\psi(\boldsymbol{T}, z)$, $\psi^*(\boldsymbol{T}, z)$ and bosonization rules

Let $r \neq 0$, and put $r(n) = \mathrm{e}^{T_{n-1}-T_n}$, where the variables T_n are defined up to a constant independent of n. We define a Hamiltonian $H_0(\boldsymbol{T}) \in \widehat{gl}(\infty)$ (all $T_n \in C$ are finite):

$$H_0(\boldsymbol{T}) := \sum_{n=-\infty}^{\infty} T_n : \psi_n^* \psi_n : \tag{3.7}$$

which produces the transformation (2.7):

$$\mathrm{e}^{\mp H_0(\boldsymbol{T})}\psi_n \mathrm{e}^{\pm H_0(\boldsymbol{T})} = \mathrm{e}^{\pm T_n}\psi_n \qquad \mathrm{e}^{\mp H_0(\boldsymbol{T})}\psi_n^* \mathrm{e}^{\pm H_0(\boldsymbol{T})} = \mathrm{e}^{\mp T_n}\psi_n^* \tag{3.8}$$

$$\mathrm{e}^{H_0(\boldsymbol{T})}|M\rangle = \mathrm{e}^{T_M - T_0}|M\rangle \tag{3.9}$$

$$\mathrm{e}^{H_0(\boldsymbol{T})}\tilde{A}(\tilde{\beta})\mathrm{e}^{-H_0(\boldsymbol{T})} = H(\tilde{\beta}) \qquad \mathrm{e}^{-H_0(\boldsymbol{T})}A(\beta)\mathrm{e}^{H_0(\boldsymbol{T})} = -H^*(\beta). \tag{3.10}$$

Let $r \neq 0$. It is convenient to consider the fermionic operators

$$\psi(\boldsymbol{T}, z) = \mathrm{e}^{H_0(\boldsymbol{T})}\psi(z)\mathrm{e}^{-H_0(\boldsymbol{T})} = \sum_{n=-\infty}^{n=+\infty} \mathrm{e}^{-T_n} z^n \psi_n \tag{3.11}$$

$$\psi^*(\boldsymbol{T}, z) = \mathrm{e}^{H_0(\boldsymbol{T})}\psi^*(z)\mathrm{e}^{-H_0(\boldsymbol{T})} = \sum_{n=-\infty}^{n=+\infty} \mathrm{e}^{T_n} z^{-n-1} \psi_n^*. \tag{3.12}$$

For the Miwa variables $\boldsymbol{t}^{\pm}(\boldsymbol{x}_{(N)})$ and $\boldsymbol{t}^{*\pm}(\boldsymbol{y}_{(N)})$, and for the 'Hamiltonians' A and $\tilde{A}$ defined by (3.1) and (3.5), one can derive the bosonization rules:

$$\mathrm{e}^{-A(\boldsymbol{t}^{*+}(\boldsymbol{y}_{(N)}))}|M\rangle = \frac{\psi(\boldsymbol{T}, y_1)\ldots\psi(\boldsymbol{T}, y_N)|M-N\rangle}{\Delta^+(M, N, \boldsymbol{T}, \boldsymbol{y}_{(N)})} \tag{3.13}$$

$$\mathrm{e}^{-A(\boldsymbol{t}^{*-}(\boldsymbol{y}_{(N)}))}|M\rangle = \frac{\psi^*(\boldsymbol{T}, y_1)\ldots\psi^*(\boldsymbol{T}, y_N)|M+N\rangle}{\Delta^-(M, N, \boldsymbol{T}, \boldsymbol{y}_{(N)})} \tag{3.14}$$

$$\langle M|\mathrm{e}^{\tilde{A}(\boldsymbol{t}^{+}(\boldsymbol{x}_{(N)}))} = \frac{\langle M-N|\psi^*(-\tilde{\boldsymbol{T}}, \frac{1}{x_N})\ldots\psi^*(-\tilde{\boldsymbol{T}}, \frac{1}{x_1})}{\tilde{\Delta}^+(M, N, \tilde{\boldsymbol{T}}, \boldsymbol{x}_{(N)})} \tag{3.15}$$

$$\langle M|\mathrm{e}^{\tilde{A}(\boldsymbol{t}^{-}(\boldsymbol{x}_{(N)}))} = \frac{\langle M+N|\psi(-\tilde{\boldsymbol{T}}, \frac{1}{x_N})\ldots\psi(-\tilde{\boldsymbol{T}}, \frac{1}{x_1})}{\tilde{\Delta}^-(M, N, \tilde{\boldsymbol{T}}, \boldsymbol{x}_{(N)})}. \tag{3.16}$$

Here $\tilde{T}_n$ are related to $\tilde{A}$ via (3.5) and $\tilde{r}(n) = \mathrm{e}^{\tilde{T}_{n-1}-\tilde{T}_n}$. The Vandermond coefficients are

$$\Delta^+(M, N, \boldsymbol{T}, \boldsymbol{y}_{(N)}) = \frac{\prod_{i<j}(y_i - y_j)}{(y_1\ldots y_N)^{N-M}} \frac{\tau(M, \boldsymbol{0}, \boldsymbol{T}, \boldsymbol{0})}{\tau(M-N, \boldsymbol{0}, \boldsymbol{T}, \boldsymbol{0})} \tag{3.17}$$

$$\Delta^-(M, N, \boldsymbol{T}, \boldsymbol{y}_{(N)}) = \frac{\prod_{i<j}(y_i - y_j)}{(y_1\ldots y_N)^{M+N}} \frac{\tau(M, \boldsymbol{0}, \boldsymbol{T}, \boldsymbol{0})}{\tau(M+N, \boldsymbol{0}, \boldsymbol{T}, \boldsymbol{0})} \tag{3.18}$$

$$\tilde{\Delta}^{\pm}(M, N, \tilde{\boldsymbol{T}}, \boldsymbol{x}_{(N)}) = (x_1\cdots x_N)\Delta^{\pm}(M, N, \tilde{\boldsymbol{T}}, \boldsymbol{x}_{(N)}). \tag{3.19}$$

The notation $\tau(M, \boldsymbol{0}, \boldsymbol{T}, \boldsymbol{0})$ is explained in the next subsection, see (3.21).

In Miwa variables one can rewrite the correlators (3.6), for instance,

$$\langle M|\mathrm{e}^{\tilde{A}(\boldsymbol{t}^{+}(\boldsymbol{x}_{(N)}))}\mathrm{e}^{-A(\boldsymbol{t}^{*+}(\boldsymbol{y}_{(N)}))}|M\rangle$$
$$= \frac{\langle M-N|\psi^*(-\tilde{\boldsymbol{T}}, \frac{1}{x_N})\ldots\psi^*(-\tilde{\boldsymbol{T}}, \frac{1}{x_1})\psi(\boldsymbol{T}, y_1)\ldots\psi(\boldsymbol{T}, y_N)|M-N\rangle}{\tilde{\Delta}^+(M, N, \tilde{\boldsymbol{T}}, \boldsymbol{x}_{(N)})\Delta^+(M, N, \boldsymbol{T}, \boldsymbol{y}_{(N)})}. \tag{3.20}$$

3.3. Toda lattice tau-function $\tau(M, \boldsymbol{t}, \boldsymbol{T}, \boldsymbol{t}^*)$

In this section we proceed with the following *Toda lattice* tau-function (2.12), which depends on the three sets of variables $\boldsymbol{t}, \boldsymbol{T}, \boldsymbol{t}^*$ and on $M \in Z$:

$$\tau(M, \boldsymbol{t}, \boldsymbol{T}, \boldsymbol{t}^*) = \langle M|\mathrm{e}^{H(\boldsymbol{t})} \exp\left(\sum_{-\infty}^{\infty} T_n : \psi_n^*\psi_n :\right) \mathrm{e}^{H^*(\boldsymbol{t}^*)}|M\rangle \tag{3.21}$$

where $:\psi_n^*\psi_n: \; = \psi_n^*\psi_n - \langle 0|\psi_n^*\psi_n|0\rangle$. Since the operator $\sum_{-\infty}^{\infty} :\psi_n^*\psi_n:$ commutes with all elements of the $\widehat{gl}(\infty)$ algebra, one can put $T_{-1} = 0$ in (3.21).

As we shall see the hypergeometric functions (1.12) and (1.8) are ratios of tau-functions (3.21) evaluated at special values of times $\boldsymbol{t}, \boldsymbol{T}, \boldsymbol{t}^*$. It is true only in the case when all parameters a_k of the hypergeometric functions are non-integers. For the case when at least one of the indices a_k is an integer, we will need a tau-function of an open Toda lattice.

The tau-function (3.21) is linear in each e^{T_n}. It is described by the following proposition:

Proposition 3. *Let functions $r(n)$ and $\tilde{r}(n)$ be defined through the relations*

$$r(n) = \mathrm{e}^{T_{n-1}-T_n} \qquad \tilde{r}(n) = \mathrm{e}^{\tilde{T}_{n-1}-\tilde{T}_n}. \tag{3.22}$$

For a tau-function (3.21) we have the expansions

$$\frac{\tau(M, \boldsymbol{t}, \boldsymbol{T}, \boldsymbol{t}^*)}{\tau(M, \mathbf{0}, \boldsymbol{T}, \mathbf{0})} = 1 + \sum_{n=1}^{+\infty} \sum_{|\boldsymbol{n}|=n} r_{\boldsymbol{n}}(M) s_{\boldsymbol{n}}(\boldsymbol{t}) s_{\boldsymbol{n}}(\boldsymbol{t}^*) \tag{3.23}$$

$$\frac{\tau(M, \boldsymbol{t}, \tilde{\boldsymbol{T}} + \boldsymbol{T}, \boldsymbol{t}^*)}{\tau(M, \mathbf{0}, \tilde{\boldsymbol{T}} + \boldsymbol{T}, \mathbf{0})} = 1 + \sum_{n=1}^{+\infty} \sum_{|\boldsymbol{n}|=n} (\tilde{r}r)_{\boldsymbol{n}}(M) s_{\boldsymbol{n}}(\boldsymbol{t}) s_{\boldsymbol{n}}(\boldsymbol{t}^*) \tag{3.24}$$

which are just the formulae (3.4) and (3.6) for functions $r \neq 0$ and $\tilde{r} \neq 0$.

We can put $\tilde{r} = 1$. The following equations hold:

$$\partial_{t_1} \partial_{t_1^*} \phi_n = r(n) \mathrm{e}^{\phi_{n-1}-\phi_n} - r(n+1) \mathrm{e}^{\phi_n - \phi_{n+1}} \qquad \mathrm{e}^{-\phi_n} = \frac{\tau_r(n+1, \boldsymbol{t}, \boldsymbol{t}^*)}{\tau_r(n, \boldsymbol{t}, \boldsymbol{t}^*)} \tag{3.25}$$

$$\big(\tau(n) := \tau_r(n, \boldsymbol{t}, \boldsymbol{t}^*)\big) \qquad \tau(n) \partial_{t_1^*} \partial_{t_1} \tau(n) - \partial_{t_1} \tau(n) \partial_{t_1^*} \tau(n) = r(n) \tau(n-1) \tau(n+1). \tag{3.26}$$

Equations (3.25) and (3.26) are still true in the case where $r(n)$ has zeros. These equations can easily be derived from (B.1) and (B.2).

If the function r has no integer zeros, using the change of variables $\varphi_n = -\phi_n - T_n$ we obtain the Toda lattice equation in the standard form (2.19) [9]. When the function r has zeros for integer values of its argument, namely $M_1 > M_2 > \cdots > M_e$, the tau-function describes a set of open Toda lattices between each pair of neighbour zeros (between neighbour zeros M_{i+1}, M_i there is an open chain with a number of sites given by $M_i - M_{i+1}$), and two semi-infinite Toda lattices, one of them ends on the smallest zero and the other on the largest zero. It follows from (3.2) and (3.4) that

$$r(M_k) = 0 \quad \Rightarrow \quad \tau_r(M_k, \boldsymbol{t}, \beta) = 1. \tag{3.27}$$

Then the Hirota equation (3.26) can be viewed as a recurrent relation which expresses the tau-function with a discrete Toda lattice variable n via $\tau_r(M_i \pm 1, \boldsymbol{t}, \beta)$, $\tau_r(M_i, \boldsymbol{t}, \beta) = 1$. Then from (3.2) and (3.4) we see the following. In the regions $M_i > M > M_{i+1}$ the series (3.4) only has a finite number of non-vanishing terms. For the region $M > M_1$ the sum is only over the Young diagrams $\boldsymbol{n}$ of length $l(\boldsymbol{n}) < M - M_1$. For the region $M < M_e$ only those diagrams $\boldsymbol{n}$ for which the conjugated diagrams $\boldsymbol{n}'$ have length $l(\boldsymbol{n}') \leqslant M_e - M$ contribute to the series (3.4).

Notation. *The notation $\tau_r(M, \boldsymbol{t}, \beta)$ will be used only for the KP tau-function (3.3), while $\tau(M, \boldsymbol{t}, \boldsymbol{T}, \boldsymbol{t}^*)$ denotes the TL tau-function (3.21). Also $\beta = \boldsymbol{t}^*$.*

3.4. Linear equations for the tau-function τ_r

Here we shall write down linear equations, which follow from the explicit fermionic representation of the tau-function (3.3) via the bosonization formulae of subsection 3.2. These equations may also be viewed as being the constraints which result in the string equations. For the variables $\boldsymbol{t}^-(\boldsymbol{x}_{(N)})$, using $\langle M|A = 0$ and using the relation $A_k = \mathrm{e}^{H_0} H_{-k} \mathrm{e}^{-H_0}$ inside the fermionic correlator (3.20), we obtain the partial differential equations for the tau-function (3.4):

$$\frac{\partial \tau_r(M, \boldsymbol{t}^-(\boldsymbol{x}_{(N)}), \boldsymbol{t}^*)}{\partial t_k^*} = \frac{1}{\tilde{\Delta}} \left(\sum_{i=1}^{N} (x_i r(-x_i \partial_{x_i}))^k \right) \tilde{\Delta} \tau_r(M, \boldsymbol{t}^-(\boldsymbol{x}_{(N)}), \boldsymbol{t}^*) \tag{3.28}$$

where $\tilde{\Delta} = \tilde{\Delta}^-(M, N, \mathbf{0}, \boldsymbol{x}_{(N)})$. These equations have the meaning of string constraint equations for the tau-function (3.4). In variables $\boldsymbol{t}^{*-}(\boldsymbol{y}_{(\infty)})$ we can rewrite (3.28) as

$$(-1)^k \sum_{i=1}^{+\infty} \frac{e_{k-1}\left(\frac{1}{y_1}, \ldots, \frac{1}{y_{i-1}}, \frac{1}{y_{i+1}}, \ldots\right)}{\prod_{j\neq i}(1 - y_i/y_j)} \frac{\partial \tau_r(M, \boldsymbol{t}^-(\boldsymbol{x}_{(N)}), \boldsymbol{t}^{*-}(\boldsymbol{y}_{(\infty)}))}{\partial y_i}$$

$$= \frac{1}{\tilde{\Delta}} \left(\sum_{i=1}^{N} (x_i r(-x_i \partial_{x_i}))^k \right) \tilde{\Delta} \tau_r(M, \boldsymbol{t}^-(\boldsymbol{x}_{(N)}), \boldsymbol{t}^{*-}(\boldsymbol{y}_{(\infty)})) \tag{3.29}$$

where $e_k(\boldsymbol{y})$ is a symmetric function defined through the relation $\prod_{i=1}^{+\infty}(1+ty_i) = \sum_{k=0}^{+\infty} t^k e_k(\boldsymbol{y})$. Also we have

$$\left(\sum_{k=1}^{M+N-1} k - \sum_{i=1}^{N} x_i \partial_{x_i} \right) \tilde{\Delta}^- \tau(M, \boldsymbol{t}^-(\boldsymbol{x}_{(N)}), \boldsymbol{T}, \boldsymbol{t}^{*-}(\boldsymbol{y}_{(N')}))\Delta^-$$

$$= \left(\sum_{k=1}^{M+N'-1} k - \sum_{i=1}^{N'} \left(\partial_{y_i} y_i\right) \right) \tilde{\Delta}^- \tau(M, \boldsymbol{t}^-(\boldsymbol{x}_{(N)}), \boldsymbol{T}, \boldsymbol{t}^{*-}(\boldsymbol{y}_{(N')}))\Delta^- \tag{3.30}$$

where $\tilde{\Delta}^- = \tilde{\Delta}^-(M, N, \mathbf{0}, \boldsymbol{x}_{(N)})$ and $\Delta^- = \Delta^-(M, N, \mathbf{0}, \boldsymbol{y}_{(N')})$. This formula is obtained by the insertion of the fermionic operator $\mathrm{res}_z : \psi^*(z) z \frac{\mathrm{d}}{\mathrm{d}z} \psi(z) :$ inside the fermionic correlator.

3.5. Determinant formulae

With the help of the Wick theorem [5], applied to (3.20), one obtains the following formulae:

Proposition 4. *A generalization of Milne's determinant formula*

$$\tau_r(M, \boldsymbol{t}^+(\boldsymbol{x}_{(N)}), \beta) = \frac{\det\left(x_i^{N-k} \tau_r(M-k+1, \boldsymbol{t}^+(x_i), \beta)\right)_{i,k=1}^{N}}{\det\left(x_i^{N-k}\right)_{i,k=1}^{N}}. \tag{3.31}$$

Proof.

$$\tau_r(M, \boldsymbol{t}^+(\boldsymbol{x}_{(N)}), \beta) = \frac{(x_1 \cdots x_N)^{N-M-1}}{\prod_{i<j}(x_i - x_j)}$$

$$\times \langle M| \psi_{M-1} \ldots \psi_{M-N} \psi^*\left(\frac{1}{x_N}\right) \ldots \psi^*\left(\frac{1}{x_1}\right) \mathrm{e}^{-A(\beta)} |M\rangle \tag{3.32}$$

$$= \frac{(x_1 \ldots x_N)^{N-M-1}}{\prod_{i<j}(x_i - x_j)} \det\left(\langle M| \psi_{M-k} \psi^*\left(\frac{1}{x_i}\right) \mathrm{e}^{-A(\beta)} |M\rangle \right)_{i,k=1}^{N} \tag{3.33}$$

$$= \frac{\det\left(x_i^{N-k} \tau_r(M-k+1, \boldsymbol{t}^+(x_i), \beta)\right)_{i,k=1}^{N}}{\det\left(x_i^{N-k}\right)_{i,k=1}^{N}}. \tag{3.34}$$

□

Proposition 5. *For* $r \neq 0$ *we have the determinant formula*

$$\tau_r(M, \boldsymbol{t}^+(\boldsymbol{x}_{(N)}), \boldsymbol{t}^{*+}(\boldsymbol{y}_{(N)})) = \frac{\det\left(F(x_i y_j)\right)_{i,j=1}^{N}}{\tilde{\Delta}^+(M, N, \mathbf{0}, \boldsymbol{x}_{(N)}) \Delta^+(M, N, \boldsymbol{T}, \boldsymbol{y}_{(N)})} \tag{3.35}$$

$$F(x_i y_j) = \langle M-N| \psi^* \left(\frac{1}{x_i}\right) \psi\left(\boldsymbol{T}, y_j\right) |M-N\rangle. \tag{3.36}$$

To prove (3.35) we consider a tau-function $\tau_r(M, \boldsymbol{t}^+(\boldsymbol{x}_{(N)}), \boldsymbol{t}^{*+}(\boldsymbol{y}_{(N)}))$ and apply Wick's theorem.

3.6. Integral representations

For the fermions (3.11) we easily obtain the following relations:

$$\int \psi(\boldsymbol{T}, \alpha z)\, \mathrm{d}\mu(\alpha) = \psi\left(\boldsymbol{T} + \boldsymbol{T}(\mu), z\right) \qquad \int \alpha^n\, \mathrm{d}\mu(\alpha) = \mathrm{e}^{-T_n(\mu)} \tag{3.37}$$

$$\int \psi^*\left(-\tilde{\boldsymbol{T}}, \frac{1}{\alpha z}\right) \mathrm{d}\tilde{\mu}(\alpha) = \psi^*\left(-\tilde{\boldsymbol{T}} - \tilde{\boldsymbol{T}}(\tilde{\mu}), \frac{1}{z}\right) \qquad \int \alpha^n\, \mathrm{d}\tilde{\mu}(\alpha) = \mathrm{e}^{-\tilde{T}_n(\tilde{\mu})} \tag{3.38}$$

where the functions μ and $\tilde{\mu}$ are some integration measures, and the shifts of times T_n are defined in terms of the moments of these measures. Therefore, thanks to the fermionic representation (3.20) we have the following relations for the tau-function:

Proposition 6. *The integral representation formula holds*

$$\begin{aligned}\int \tilde{\Delta}_{\tilde{\boldsymbol{T}}}(\tilde{\alpha}\boldsymbol{x}_{(N)}) &\frac{\tau(M, \boldsymbol{t}^+(\tilde{\alpha}\boldsymbol{x}_{(N)}), \boldsymbol{T} + \tilde{\boldsymbol{T}}, \boldsymbol{t}^+(\boldsymbol{\alpha}\boldsymbol{y}_{(N)}))}{\tau(M, \boldsymbol{0}, \boldsymbol{T} + \tilde{\boldsymbol{T}}, \boldsymbol{0})} \Delta_{\boldsymbol{T}}(\boldsymbol{\alpha}\boldsymbol{y}_{(N)}) \prod_{i=1}^N \mathrm{d}\tilde{\mu}(\tilde{\alpha}_i) \prod_{i=1}^N \mathrm{d}\mu(\alpha_i) \\ &= \tilde{\Delta}_{\tilde{\boldsymbol{T}}+\tilde{\boldsymbol{T}}(\tilde{\mu})}(\boldsymbol{x}_{(N)}) \frac{\tau(M, \boldsymbol{t}^+(\boldsymbol{x}_{(N)}), \boldsymbol{T} + \tilde{\boldsymbol{T}} + \tilde{\boldsymbol{T}}(\tilde{\mu}) + \boldsymbol{T}(\mu), \boldsymbol{t}^+(\boldsymbol{y}_{(N)}))}{\tau(M, \boldsymbol{0}, \boldsymbol{T} + \tilde{\boldsymbol{T}} + \tilde{\boldsymbol{T}}(\tilde{\mu}) + \boldsymbol{T}(\mu), \boldsymbol{0})} \\ &\quad \times \Delta_{\boldsymbol{T}+\boldsymbol{T}(\mu)}(\boldsymbol{y}_{(N)})\end{aligned} \tag{3.39}$$

where $\Delta_{\boldsymbol{T}}(\boldsymbol{\alpha}\boldsymbol{y}_{(N)}) = \Delta^+(M, N, \boldsymbol{T}, \boldsymbol{\alpha}\boldsymbol{y}_{(N)})$, $\tilde{\Delta}_{\tilde{\boldsymbol{T}}}(\tilde{\alpha}\boldsymbol{x}_{(N)}) = \tilde{\Delta}^+(M, N, \tilde{\boldsymbol{T}}, \tilde{\alpha}\boldsymbol{x}_{(N)})$, $\boldsymbol{\alpha}\boldsymbol{y}_{(N)} = (\alpha_1 y_1, \alpha_2 y_2, \ldots, \alpha_N y_N)$ *and* $\tilde{\boldsymbol{\alpha}}\boldsymbol{x}_{(N)} = (\tilde{\alpha}_1 x_1, \tilde{\alpha}_2 x_2, \ldots, \tilde{\alpha}_N x_N)$. *In particular,*

$$\begin{aligned}\int \frac{\tau\left(M, \boldsymbol{t}, \boldsymbol{T}, \boldsymbol{t}^+(\boldsymbol{\alpha}\boldsymbol{y}_{(N)})\right)}{\tau\left(M, \boldsymbol{0}, \boldsymbol{T}, \boldsymbol{0}\right)} &\Delta_{\boldsymbol{T}}(\boldsymbol{\alpha}\boldsymbol{y}_{(N)}) \prod_{i=1}^N \mathrm{d}\mu(\alpha_i) \\ &= \frac{\tau\left(M, \boldsymbol{t}, \boldsymbol{T} + \boldsymbol{T}(\mu), \boldsymbol{t}^+(\boldsymbol{y}_{(N)})\right)}{\tau\left(M, \boldsymbol{0}, \boldsymbol{T} + \boldsymbol{T}(\mu), \boldsymbol{0}\right)} \Delta_{\boldsymbol{T}+\boldsymbol{T}(\mu)}(\boldsymbol{y}_{(N)}).\end{aligned} \tag{3.40}$$

Remember that an arbitrary linear combination of tau-functions is not a tau-function. Formulae (3.39) and (3.40) give the integral representations for the tau-function (3.3). It may help to express a tau-function with the help of a simpler one.

Let us consider the following q-integrals [10]:

$$-q^{-a-1} \int_0^\infty \psi(\boldsymbol{T}, \alpha(q^{-1} - 1)z) E_q^{-1}(-\alpha)\alpha^a\, \mathrm{d}_q\alpha = \psi\left(\boldsymbol{T} + \boldsymbol{T}^a, z\right) \tag{3.41}$$

$$\frac{1}{\Gamma_q(b-a)} \int_0^1 \psi(\boldsymbol{T}, \alpha z)\alpha^a \frac{(\alpha q; q)_\infty}{(\alpha q^{b-a}; q)_\infty}\, \mathrm{d}_q\alpha = \psi(\boldsymbol{T} + \boldsymbol{T}^{a,b}, z) \tag{3.42}$$

where $E_q(x) = ((1-q)x; q)_\infty^{-1}$. Then

$$T_n^a = -\log\left((1-q)^n \Gamma_q(a+n+1)\right) \qquad T_n^{a,b} = \log \frac{\Gamma_q(b+n+1)}{\Gamma_q(a+n+1)}. \tag{3.43}$$

In the same way one can consider the Miwa change to $\boldsymbol{t}^-$ of (1.7).

Now we are able to write down integration formulae for Milne hypergeometric functions, see examples 2, 3 and (3.24), (3.20). We can express ${}_{p+1}\Phi_s$ and ${}_{p+1}\Phi_{s+1}$ in terms of ${}_p\Phi_s$ with the help of (3.41) and (3.42). In [1] a different integral representation formula was presented, which was based on the q-analogue of Selberg's integral of Askey and Kadell.

3.7. Specification of function r

Example 1. Let $r(n) = 1$ for all n,

$$\tau_{r=1}(M, \boldsymbol{t}, \boldsymbol{t}^*) = \exp\left(\sum_{n=1}^{\infty} n t_n t_n^*\right) \tag{3.44}$$

which is a vacuum tau-function for the two-dimensional Toda lattice. Formula (3.44) is a manifestation of the summation formulae for the Schur functions [10]. Let us note that this is also an example of the function ${}_1\Phi_0$ (see (1.13)).

Example 2.

$$_p r_s^{(q)}(n) = \frac{\prod_{i=1}^{p}(1 - q^{a_i+n})}{\prod_{i=1}^{s}(1 - q^{b_i+n})}. \tag{3.45}$$

For the variables $\boldsymbol{t}^+(\boldsymbol{x}_{(N)})$ and setting $\boldsymbol{t}^* = \boldsymbol{t}^{*+}(1, q, q^2, \ldots)$ we obtain Milne's hypergeometric function (1.8),

$$\begin{aligned}\tau_r(M, \boldsymbol{t}^+(\boldsymbol{x}_{(N)}), \boldsymbol{t}^{*+}(1, q, q^2, \ldots)) &= {}_p\Phi_s\left(\begin{matrix} a_1 + M, \ldots, a_p + M \\ b_1 + M, \ldots, b_s + M \end{matrix}\middle| q, \boldsymbol{x}_{(N)}\right) \\ &= \sum_{\substack{\boldsymbol{n} \\ l(\boldsymbol{n})\leqslant N}} \frac{(q^{a_1+M}; q)_{\boldsymbol{n}} \cdots (q^{a_p+M}; q)_{\boldsymbol{n}}}{(q^{b_1+M}; q)_{\boldsymbol{n}} \cdots (q^{b_s+M}; q)_{\boldsymbol{n}}} \frac{q^{n(\boldsymbol{n})}}{H_{\boldsymbol{n}}(q)} s_{\boldsymbol{n}}(\boldsymbol{x}_{(N)}).\end{aligned} \tag{3.46}$$

Example 3. To obtain Milne's hypergeometric function of two sets of variables $\boldsymbol{x}_{(N)}, \boldsymbol{y}_{(N)}$ we use $\boldsymbol{t}^+(\boldsymbol{x}_{(N)})$ and $\boldsymbol{t}^{*+}(\boldsymbol{y}_{(N)})$. This choice restricts the sum over partitions $\boldsymbol{n}$ with $l(\boldsymbol{n}) \leqslant N$. Put

$$_p r_s^{(q)}(n) = \frac{\prod_{i=1}^{p}(1 - q^{a_i+n})}{\prod_{i=1}^{s}(1 - q^{b_i+n})} \frac{1}{1 - q^{N-M+n}} \tag{3.47}$$

$$\mathrm{e}^{T_n} = (1 - q)^n \Gamma_q(n + N - M + 1) \frac{\prod_{i=1}^{s}(1 - q)^n \Gamma_q(b_i + n + 1)}{\prod_{i=1}^{p}(1 - q)^n \Gamma_q(a_i + n + 1)} \tag{3.48}$$

$$\Gamma_q(a) = (1 - q)^{1-a} \frac{(q; q)_\infty}{(q^a, q)_\infty} \qquad (q^a, q)_n = (1 - q)^n \frac{\Gamma_q(a + n)}{\Gamma_q(a)}. \tag{3.49}$$

Here $\Gamma_q(a)$ is a q-deformed Gamma-function. We obtain (see section 3 of [12] for help) Milne's formula (1.13)

$$\begin{aligned}\tau_r(M, \boldsymbol{t}^+(\boldsymbol{x}_{(N)}), \boldsymbol{t}^{*+}(\boldsymbol{y}_{(N)})) &= {}_p\Phi_s\left(\begin{matrix} a_1 + M, \ldots, a_p + M \\ b_1 + M, \ldots, b_s + M \end{matrix}\middle| q, \boldsymbol{x}_{(N)}, \boldsymbol{y}_{(N)}\right) \\ &= \sum_{\substack{\boldsymbol{n} \\ l(\boldsymbol{n})\leqslant N}} \frac{q^{n(\boldsymbol{n})}}{H_{\boldsymbol{n}}(q)} \frac{s_{\boldsymbol{n}}(\boldsymbol{x}_{(N)}) s_{\boldsymbol{n}}(\boldsymbol{y}_{(N)})}{s_{\boldsymbol{n}}(1, q, \ldots, q^{N-1})} \frac{(q^{a_1+M}; q)_{\boldsymbol{n}} \cdots (q^{a_p+M}; q)_{\boldsymbol{n}}}{(q^{b_1+M}; q)_{\boldsymbol{n}} \cdots (q^{b_s+M}; q)_{\boldsymbol{n}}}.\end{aligned} \tag{3.50}$$

This is the KP tau-function (but not the TL one because (3.47) depends on the TL variable M). To find the basic hypergeometric function of one set of variables ${}_p\Phi_s\left(\begin{smallmatrix} a_1+M, \ldots, a_p+M \\ b_1+M, \ldots, b_s+M \end{smallmatrix}\middle| q, \boldsymbol{x}_{(N)}\right)$ we must put indeterminates $\boldsymbol{y}_{(N)}$ in (3.50) as $y_i = q^{i-1}, i = (1, \ldots, N)$. This function satisfies the following q-difference equation:

$$\left(\frac{1}{x}\left(1 - q^D\right) - {}_p r_s^{(q)}(D)\right) {}_p\Phi_s(a_1, \ldots, a_p; b_1, \ldots, b_s; q, x) = 0 \qquad D := x\partial_x. \tag{3.51}$$

Example 4. The hypergeometric function (1.13), (3.50), ${}_1\Phi_1\left(\begin{smallmatrix}a\\b\end{smallmatrix}\middle|q, \boldsymbol{x}_{(N)}, \boldsymbol{y}_{(N)}\right)$ can be degenerated to ${}_1\Phi_0\big(a\big|q, \boldsymbol{x}_{(N)}, \boldsymbol{y}_{(N)}\big)$ by taking $b \to +\infty$ (remember that $|q| < 1$). The limit $b \to -\infty$ (with the rescaling of times $x_i, y_i \to q^{b/2}x_i, q^{b/2}x_i$) is also of interest. Consider this limit and put $a = N - M$.

Now we obtain an example of the KP tau-function (3.4) which is not a hypergeometric function. Take $T_n = -\frac{\gamma}{2}(n+\frac{1}{2})^2$, or equivalently

$$r(n) = q^{-n} \qquad q = \mathrm{e}^{-\gamma} \tag{3.52}$$

and rescale the times once more: $t_k = \alpha^k p_k, t_k^* = \alpha^k p_k^*$. We obtain the series

$$\langle M|\mathrm{e}^{H(\boldsymbol{t})}\mathrm{e}^{A(\boldsymbol{t}^*)}|M\rangle = \sum_{\boldsymbol{n}} \alpha^{|\boldsymbol{n}|}\mathrm{e}^{\gamma f_2(\boldsymbol{n})} s_{\boldsymbol{n}}(\boldsymbol{p}) s_{\boldsymbol{n}}(\boldsymbol{p}^*) \tag{3.53}$$

$$f_2(\boldsymbol{n}) = \tfrac{1}{2}\sum_i \left[(n_i - i + M - \tfrac{1}{2})^2 - (-i + M - \tfrac{1}{2})^2\right] \tag{3.54}$$

which was recently considered in [20] (our notation $\boldsymbol{n}, \alpha, \gamma$ is related to λ, q, β in [20], respectively). This series is a generating function for double Hurwitz numbers $Hur_{d,b}(n, m)$ introduced in [20]. The formula presented in [20] in our terms reads as

$$\log\langle M|\mathrm{e}^{H(\boldsymbol{t})}\mathrm{e}^{A(\boldsymbol{t}^*)}|M\rangle = \sum_{d,b,n,m} \alpha^d \gamma^b p_n p_m^* Hur_{d,b}(n, m)/b! \tag{3.55}$$

for A see (3.1) and (3.52). Therefore, the generating function for the double Hurwitz numbers is expressed in terms of a group cocycle of the ΨDO on the circle (see appendix 2), which is the fermionic correlator under the logarithm.

Example 5. Notation and notions for this example are borrowed from [16]. Let r be a rational function of Jakoby theta-functions $\theta(2x\eta|\tilde{\tau})$, where $\tilde{\tau}$ is an elliptic modulus:

$${}_pr_s^{(\eta)}(n) = \frac{\prod_{i=1}^p \theta(2\eta(a_i+n)|\tilde{\tau})}{\theta(2\eta(N-M+n)|\tilde{\tau})\prod_{i=1}^s \theta(2\eta(b_i+n)|\tilde{\tau})} \qquad \mathrm{e}^{T_{n-1}} = \frac{[N-M]_n \prod_{i=1}^s [b_i]_n}{\prod_{i=1}^p [a_i]_n}. \tag{3.56}$$

Here the elliptic Pochhammer symbol $[a]_n$ is defined in terms of the elliptic number $[a]$,

$$[a] = \theta(2a\eta|\tilde{\tau}) \qquad [a]_k = [a][a+1][a+2]\cdots[a+k-1]. \tag{3.57}$$

One can associate the elliptic Pochhammer symbol with a given partition $\boldsymbol{n}$:

$$[a]_{\boldsymbol{n}} = [a]_{n_1}[a-1]_{n_2}\cdots[a-l+1]_{n_l}. \tag{3.58}$$

For the variables $\boldsymbol{t}^+(\boldsymbol{x}_{(N)})$ and $\boldsymbol{t}^{*+}(\boldsymbol{y}_{(N)})$ we can introduce the hypergeometric function

$$\begin{aligned}\langle M|\mathrm{e}^{H(\boldsymbol{t}^+(\boldsymbol{x}_{(N)}))}\mathrm{e}^{A(\boldsymbol{t}^{*+}(\boldsymbol{y}_{(N)}))}|M\rangle &= {}_pF_s^{(\eta)}\left(\begin{matrix}a_1+M,\ldots,a_p+M\\ b_1+M,\ldots,b_s+M\end{matrix}\middle|\eta, \boldsymbol{x}_{(N)}, \boldsymbol{y}_{(N)}\right)\\ &= \sum_{\substack{\boldsymbol{n}\\ l(\boldsymbol{n})\leqslant N}} \frac{s_{\boldsymbol{n}}(\boldsymbol{x}_{(N)})s_{\boldsymbol{n}}(\boldsymbol{y}_{(N)})}{[N]_{\boldsymbol{n}}} \frac{[a_1+M]_{\boldsymbol{n}}\cdots[a_p+M]_{\boldsymbol{n}}}{[b_1+M]_{\boldsymbol{n}}\cdots[b_s+M]_{\boldsymbol{n}}}.\end{aligned} \tag{3.59}$$

As in the case of (1.13) this is the KP tau-function which is not the TL tau-function because the factor $[N]_{\boldsymbol{n}}$ in the denominator does not depend on M. For $N = 1$ we obtain an elliptic hypergeometric function of one variable [16]. For instance, to obtain the elliptic very-well-poised hypergeometric function

$${}_{p+1}W_p(\alpha_1; \alpha_4, \alpha_5, \ldots, \alpha_{p+1}; z|\eta, \tau) = \sum_{n=0}^{\infty} z^n \frac{[\alpha_1+2n][\alpha_1]_n}{[\alpha_1][n]!}\prod_{m=1}^{p-2}\frac{[\alpha_{m+3}]_n}{[\alpha_1-\alpha_{m+3}+1]_n} \tag{3.60}$$

we choose

$$t_n = \frac{z^n}{n} \qquad t_n^* = \frac{1}{n} \qquad \mathrm{e}^{T_{n-1}} = \frac{[\alpha_1][n]!}{[\alpha_1 + 2n][\alpha_1]_n} \prod_{m=1}^{p-2} \frac{[\alpha_1 - \alpha_{m+3} + 1]_n}{[\alpha_{m+3}]_n}. \tag{3.61}$$

3.8. Different representations

Let us rewrite the hypergeometric series in a different way representing all Pochhammer coefficients $(q^a; q)_n$ and $(a)_n$ through Schur functions. This gives us the opportunity to interchange the role of the Pochhammer coefficients and Schur functions in (1.8) and (1.13), and to present different fermionic representations of the hypergeometric functions. We have the relations (see [12])

$$\prod_{(i,j)\in n} (1 - q^{a+j-i}) = \frac{s_n(\boldsymbol{t}(a, q))}{s_n(\boldsymbol{t}(+\infty, q))} \tag{3.62}$$

where the parameters $t_m(a, q)$ are chosen via a generalized Miwa transform with a parameter ('multiplicity') a:

$$\begin{aligned} t_m(a, q) &= \frac{1 - (q^a)^m}{m(1 - q^m)} \qquad m = (1, 2, \ldots) \\ s_n(\boldsymbol{t}(+\infty, q)) &= \lim_{a\to+\infty} s_n(\boldsymbol{t}(a, q)) = \frac{q^{n(n)}}{H_n(q)}. \end{aligned} \tag{3.63}$$

Now we rewrite the series (3.46) only in terms of Schur functions:

$$\begin{aligned} {}_p\Phi_s &\left(\begin{matrix} a_1 + M, \ldots, a_p + M \\ b_1 + M, \ldots, b_s + M \end{matrix} \middle| q, \boldsymbol{x}_{(N)}, \boldsymbol{y}_{(N)} \right) = \tau_r(M, \boldsymbol{t}(+\infty, q), \boldsymbol{t}^*) \\ &= \sum_{\substack{n \\ l(n)\leqslant N}} \frac{\prod_{k=1}^p s_n(\boldsymbol{t}(a_k + M, q))}{\prod_{k=1}^s s_n(\boldsymbol{t}(b_k + M, q))} (s_n(\boldsymbol{t}(+\infty, q)))^{s-p+1} \frac{s_n(\boldsymbol{x}_{(N)}) s_n(\boldsymbol{y}_{(N)})}{s_n(\boldsymbol{t}(N, q))}. \end{aligned} \tag{3.64}$$

A nice feature of this formula is that there are no number coefficients at all, it is a sum of ratios of Schur functions only.

We obtain different fermionic representations of hypergeometric functions (3.64), and they are parametrized by a non-integer complex number b:

Proposition 7. *For* $r = {}_pr_s^{(q)}$ *(see (3.45)) we have*

$$\tau_r(M, \boldsymbol{t}(+\infty, q), \boldsymbol{t}^*) = \tau_{r_b}(M, \boldsymbol{t}(b + M, q), \boldsymbol{t}^*) \qquad r_b(n) = \frac{r(n)}{1 - q^{b+n}}. \tag{3.65}$$

Conclusion

We obtain Milne's hypergeometric functions as certain tau-functions of the KP hierarchy. The functions (1.8) are also the TL tau-functions. It means that we have a set of new relations on the multivariate hypergeometric functions. For instance, all hypergeometric functions of the form (3.3) satisfy bilinear Hirota equations [5] of the KP theory. One can obtain the fermionic representations for different special functions and polynomials related to these hypergeometric functions. Using an integral representation one can express hypergeometric functions as the integral of rather simple hypergeometric function. We also obtain determinant representation of (1.13), which may allow us to analyse analytical properties of multivariate tau-functions in terms of functions of only one variable. We wrote down the system of linear equations on the tau-function (1.13), which may allow us to find applications to quantum mechanical problems. Let us note that we obtain a q-deformed version of these hypergeometric functions as tau-functions not of a q-deformed KP hierarchy [17], but of the usual KP hierarchy. It is now an interesting problem to establish links between these results and a group-theoretic approach to the q-special functions [10, 11] and matrix integrals.

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Appendix A. Orthogonal polynomials and matrix integrals

It is known that the hypergeometric functions (1.8) and (1.13) for $q \to 1$ appear in the group representation theory and are connected with the so-called matrix integrals [11]. On the other hand, the set of examples [15] reveals a connection between the matrix integrals [19] and the soliton theory. To establish this connection in our case, it is useful to consider the related systems of the orthogonal polynomials. Let us briefly describe how to write down these polynomials.

Let M_+ (respectively, M_-) be the largest (respectively, the smallest) integer zero of $r(n)$. The function

$$f_r^+(zz^*) = \sum_{n=0}^{+\infty}(zz^*)^{n+M_+}\mathrm{e}^{T_{n+M_+}-T_{M_+}} \tag{A.1}$$

is the eigenfunction of the operator $\frac{1}{z}r(D)$, $D = z\frac{\mathrm{d}}{\mathrm{d}z}$ with the eigenvalue z^*. Since operator $r(D)$ is invertible on the functions $\{z^M, M > M_+\}$ we write

$$f_r^+(zz^*) = \left(1 - \frac{1}{r(D)}zz^*\right)^{-1}(zz^*)^{M_+}. \tag{A.2}$$

For example, if we take $r(n) = n$ we obtain $f_r^+(zz^*) = \mathrm{e}^{zz^*}$.

We use these functions as weight functions for a system of orthogonal polynomials $\{\pi_n^\pm, n = 0, 1, 2, \ldots\}$, related to the hypergeometric solution of KP:

$$\int_\gamma \pi_n^-(\boldsymbol{t}, \beta, z)\mathrm{e}^{\xi(\boldsymbol{t},z)} f_r^+(zz^*)\mathrm{e}^{\xi(\beta,z^*)}\pi_m^+(\boldsymbol{t}, \beta, z^*)\,\mathrm{d}z\,\mathrm{d}z^* = \mathrm{e}^{-\phi_{M_++n}(\boldsymbol{t},\beta)}\delta_{n,m}. \tag{A.3}$$

For $r(n) = n$ we have the corresponding two matrix integral [19]:

$$\tau(M, \boldsymbol{t}, \beta) = \int \mathrm{e}^{Tr\xi(\boldsymbol{t},Z)} f_r^+\left(Tr(ZZ^*)\right)\mathrm{e}^{Tr\xi(\beta,Z^*)}\,\mathrm{d}Z\,\mathrm{d}Z^*. \tag{A.4}$$

Here Z, Z^* are Hermitian $M \times M$ matrices.

Appendix B. String equations and ΨDO on the circle

Let us describe relevant string equations following Takasaki and Takebe [3]. We shall also consider this topic in a more detailed paper.

Following [9] we introduce infinite matrices to describe KP and TL flows and symmetries. Zakharov–Shabat dressing matrices are K and $\bar{K}$. K is a lower triangular matrix with unit main diagonal: $(K)_{ii} = 1$. $\bar{K}$ is an upper triangular matrix. The matrices $K, \bar{K}$ depend on the parameters $M, \boldsymbol{t}, \boldsymbol{T}, \boldsymbol{t}^*$. The matrices $(\Lambda)_{ik} = \delta_{i,k-1}$, $(\bar{\Lambda})_{ik} = \delta_{i,k+1}$. For each value of $\boldsymbol{t}, \boldsymbol{T}, \boldsymbol{t}^*$ and $M \in Z$ they solve a Gauss (Riemann–Hilbert) factorization problem for infinite matrices:

$$\bar{K} = KG(M, \boldsymbol{t}, \boldsymbol{T}, \boldsymbol{t}^*) \qquad G(M, \boldsymbol{t}, \boldsymbol{T}, \boldsymbol{t}^*) = \mathrm{e}^{\xi(\boldsymbol{t},\Lambda)}\Lambda^M G(\boldsymbol{0}, \boldsymbol{T}, \boldsymbol{0})\bar{\Lambda}^M \mathrm{e}^{\xi(\boldsymbol{t}^*,\bar{\Lambda})}. \tag{B.1}$$

We put $\log(\bar{K}_{ii}) = \phi_{i+M}$ and a set of fields $\phi_i(\boldsymbol{t}, \boldsymbol{t}^*)$, $(-\infty < i < +\infty)$ solves the hierarchy of higher two-dimensional TL equations. Take $L = K\Lambda K^{-1}$, $\bar{L} = \bar{K}\bar{\Lambda}\bar{K}^{-1}$ and $(\Delta)_{ik} = \mathrm{i}\delta_{i,k}$, $\widehat{M} = K\Delta K^{-1} + M + \sum nt_n L^n$, $\widehat{\bar{M}} = \bar{K}\Delta\bar{K}^{-1} + M + \sum nt_n^*\bar{L}^n$. Then the KP additional symmetries [3, 6] and higher TL flows [9] are written as

$$\partial_{\beta_n} K = -\left(\left(r(\widehat{M})L^{-1}\right)^n\right)_- K \qquad \partial_{\beta_n}\bar{K} = \left(\left(r(\widehat{M})L^{-1}\right)^n\right)_+ \bar{K} \tag{B.2}$$

$$\partial_{t_n^*} K = -\left(\bar{L}^n\right)_- K \qquad \partial_{t_n^*}\bar{K} = \left(\bar{L}^n\right)_+ \bar{K}. \tag{B.3}$$

Then the string equations are

$$\bar{L}L = r(\widehat{M}) \qquad \widehat{\bar{M}} = \widehat{M}. \tag{B.4}$$

The first equation is a manifestation of the fact that the group time β_1 of the additional symmetry of KP can be identified with the Toda lattice time t_1^*. In terms of the tau-function written in Miwa variables we have equations (3.28). The second string equation is related to the symmetry of our tau-functions with respect to $\boldsymbol{t} \leftrightarrow \beta$. The KP tau-function (3.4) can be obtained as follows:

$$G(M, \boldsymbol{t}, \boldsymbol{T}, \boldsymbol{t}^*) = G(\boldsymbol{0}, \boldsymbol{T}, \boldsymbol{0})U(M, \boldsymbol{t}, \beta) \qquad U(M, \boldsymbol{t}, \beta) = U^+(\boldsymbol{t})U^-(M, \beta). \tag{B.5}$$

$$U^+(\boldsymbol{t}) = \exp\left(\xi(\boldsymbol{t}, \Lambda)\right) \qquad U^-(M, \beta) = \exp\left(\xi(\beta, \Lambda^{-1}r\,(\Delta + M))\right). \tag{B.6}$$

The matrix $G(\boldsymbol{0}, \boldsymbol{T}, \boldsymbol{0})$ is related to the transformation of equation (2.19) to equation (3.25). By taking the projection [9] $U \mapsto U_{--}$ for non-positive values of matrix indices we obtain a determinant representation of the tau-function (3.4):

$$\tau_r(M, \boldsymbol{t}, \beta) = \frac{\det U_{--}(M, \boldsymbol{t}, \beta)}{\det\left(U_{--}^+(\boldsymbol{t})\right)\det\left(U_{--}^-(M, \beta)\right)} = \det U_{--}(M, \boldsymbol{t}, \beta) \tag{B.7}$$

since both determinants in the denominator are equal to unity. Formula (B.7) is also a Segal–Wilson formula for the $GL(\infty)$ 2-cocycle $C_M\left(U^+(-\boldsymbol{t}), U^-(-\beta)\right)$ ('Japanese cocycle' [5]). Choosing the function r as in subsection 3.7 we obtain the hypergeometric functions listed in section 1.

Remark 2. Therefore, the hypergeometric functions which were considered above have the meaning of a $GL(\infty)$ two-cocycle on the two multiparametrical group elements $U^+(\boldsymbol{t})$ and $U^-(M, \beta)$. Both elements $U^+(\boldsymbol{t})$ and $U^-(M, \beta)$ can be considered as elements of a group of pseudodifferential operators on the circle. The corresponding Lie algebras consist of the multiplication operators $\{z^n; n \in N_0\}$ and of the pseudodifferential operators $\{\left(\frac{1}{z}r(z\frac{\mathrm{d}}{\mathrm{d}z} + M)\right)^n; n \in N_0\}$. Two sets of group times $\boldsymbol{t}$ and β play the role of indeterminates of the hypergeometric functions (1.13). Formulae (3.4) and (3.6) mean the expansion of a $GL(\infty)$ group 2-cocycle in terms of a corresponding Lie algebra 2-cocycle

$$c_M\left(z, \frac{1}{z}r(D)\right) = r(M) \qquad c_M\left(\tilde{r}(D)z, \frac{1}{z}r(D + M)\right) = \tilde{r}(M)r(M) \qquad D = z\frac{\mathrm{d}}{\mathrm{d}z}. \tag{B.8}$$

A Japanese cocycle is cohomological to the Khesin–Kravchenko cocycle [18] for the ΨDO on the circle:

$$c_M \sim c_0 \sim \omega_M \tag{B.9}$$

which is

$$\omega_M(A, B) = \oint \mathrm{res}_\partial\, A[\log(D+M), B]\,\mathrm{d}z \qquad A, B \in \Psi\mathrm{DO}. \tag{B.10}$$

For the group cocycle we have

$$C_M\left(\mathrm{e}^{-\sum z^n t_n}, \mathrm{e}^{-\sum (z^{-1}r(D))^n \beta_n}\right) = \tau_r(M, \boldsymbol{t}, \beta) \tag{B.11}$$

where we imply that the order of $\Psi\mathrm{DO}r(D)$ is 1 or less. Let us also note that in the case of hypergeometric functions ${}_p\Phi_s$ (1.8), the condition $p - s \leqslant 1$ is the condition for the convergence of this hypergeometric series (see [10]). Namely, the radius of convergence is finite in the case where $p - s = 1$, it is infinite when $p - s < 1$ and it is zero for $p - s > 1$.

Remark 3. Let $D = z\frac{\mathrm{d}}{\mathrm{d}z}$. Consider

$$\begin{aligned} w(n, z) &= \exp\left[-\sum_{m=0}^{\infty} t_m^* \left(\frac{1}{z} r(D)\right)^m\right] z^n \\ w^*(n, z) &= \exp\left[\sum_{m=0}^{\infty} t_m^* \left(\frac{1}{z} r(-D)\right)^m\right] z^{-n}\,\mathrm{d}z. \end{aligned} \tag{B.12}$$

The set of functions $\{w(n, z), n = M, M+1, M+2, \ldots\}$, may be identified with a Sato Grassmannian related to the cocycle (B.11). The dual Grassmannian is the set of 1-forms $\{w^*(n, z), n = M, M+1, M+2, \ldots\}$.

References

[1] Milne S C 1992 Summation theorems for basic hypergeometric series of Schur function argument *Progress in Approximation Theory* ed A A Gonchar and E B Saff (New York: Springer) pp 51–77

[2] Takasaki K 1984 Initial value problem for the Toda lattice hierarchy *Adv. Stud. Pure Math.* **4** 139–63

[3] Takasaki K 1996 The Toda lattice hierarchy and generalized string equation *Commun. Math. Phys.* **181** 131
(Takasaki K 1995 *Preprint* hep-th/9506089)
Takebe T and Takasaki K 1995 *Rev. Math. Phys.* **7** 743–808
(Takebe T and Takasaki K 1994 *Preprint* hep-th/94050096)

[4] Zakharov V E and Shabat A B 1974 *J. Funct. Anal. Appl.* **8** 226
Zakharov V E and Shabat A B 1979 *J. Funct. Anal. Appl.* **13** 166

[5] Jimbo M and Miwa T 1983 *Publ. RIMS Kyoto Univ.* **19** 943
Date E, Jimbo M, Kashiwara M and Miwa T 1983 Transformation groups for soliton equations *Nonlinear Integrable Systems—Classical Theory and Quantum Theory* ed M Jimbo and T Miwa (Singapore: World Scientific) pp 39–120

[6] Dickey L A 1991 *Soliton Equations and Hamiltonian System* (Singapore: World Scientific)

[7] Zakharov V E, Manakov S V, Novikov S P and Pitaevsky L P 1980 *The Theory of Solitons. The Inverse Scattering Method* (Moscow: Nauka)

[8] Mikhailov A V 1979 On the integrability of two-dimensional generalization of the Toda lattice *Lett. J. Exp. Theor. Phys.* **30** 443–8

[9] Ueno K and Takasaki K 1984 *Adv. Stud. Pure Math.* **4** 1–95

[10] Vilenkin N Ya and Klimyk A U 1995 *Representation of Lie groups and Special Functions* (Dordrecht: Kluwer)

[11] Vilenkin N Ya and Klimyk A U 1992 *Representation of Lie Groups and Special Functions* vol 3 *Classical and Quantum Groups and Special Functions* (Dordrecht: Kluwer)

[12] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* (Oxford: Clarendon)

[13] Orlov A Yu 1988 Vertex operators, ∂ bar problem, symmetries, Hamiltonian and Lagrangian formalism of (2+1) dimensional integrable systems *Proc. 3rd Kiev. Int. Workshop (1987)* vol I (Singapore: World Scientific) pp 116–34

[14] Morozov A Yu and Vinet L 1993 q-hypergeometric functions in the formalism of free fields *Mod. Phys. Lett.* A **8** 2891–902

[15] Kontsevich M 1991 *Funk. Anal. ego Prilozh.* **25** no 2 50–7
Gerasimov A, Marshakov A, Mironov A, Morozov A and Orlov A 1991 *Nucl. Phys.* B 565

[16] Zabrodin A 1999 Commuting difference operators with elliptic coefficients from Baxter's vacuum vectors *Preprint* QA/9912218
Frenkel I and Turaev V 1997 Elliptic solutions of the Yang–Baxter equations and modular hypergeometric functions *The Arnold–Gelfand Mathematical Seminars* (Boston, MA: Birkhäuser) pp 171–204
Zhedanov A and Spiridonov V 2000 *Commun. Math. Phys.* **210** 49–83

[17] Bogdanov L V 1995 *Physica* D **87** 58–63
(Bogdanov L V 1995 *Preprint* q-alg/9501031)
Mironov A D 1998 *Theor. Math. Phys.* **114** 127
(Mironov A D 1997 *Preprint* q-alg/9711006)
Tu Ming-Hsien 1999 *Lett. Math. Phys.* **49** 95–103

[18] Kravchenko O and Khesin B 1991 *Funct. Anal. Appl.* **25** 83

[19] Mehta M L 1991 *Random Matrices* (New York: Academic)

[20] Okounkov A 2000 Toda equations for Hurwitz numbers *Preprint* math. AG/0004128